Effects of Air Drag on Near-Circular Satellite Orbits

J. OTTERMAN* AND K. LICHTENFELD†
General Electric Company, Philadelphia, Pa.

This paper is concerned with satellite trajectories subject to the influence of air drag when the satellite is launched into an orbit with very small eccentricity. In the first section of the paper, the general differential and integral equations for decay of circular and near-circular orbits are derived. The density appears as an unspecified function of travel angle. The next section deals with spherically symmetrical atmosphere. The case of constant density is solved, showing the oscillatory nature of the solutions. The solution is given in its more general form, which includes nonzero initial conditions and which shows that the oscillatory terms depend very strongly on the initial velocity, i.e., the initial departure from an exactly circular orbit. Next, the case of large descent into exponentially increasing atmosphere is solved. The solution includes the oscillatory terms that tend to die out with the descent. In the last section, the effects of diurnal variations and atmospheric oblateness are treated. It is shown that effects of diurnal variations can be pronounced, but that the effects of atmospheric oblateness are relatively small. Numerical examples are included.

Introduction

THE subject of this paper is the analysis of satellite trajectories subject to the influence of air drag. The satellite is assumed to be launched into an orbit with very small eccentricity.

Orbital perturbations can be divided into three classes: 1) short-periodic when they have characteristic periods smaller than or equal to the period of revolution, 2) long-periodic when they have characteristic periods that are long compared with the period of revolution, and 3) secular when they monotonically increase with time.[1] Although generally the secular and long-periodic effects are of greatest interest, short-periodic perturbations must be taken into account in orbit-computing programs.[1] The importance of considering these perturbations increases with increased precision of orbit determination made possible by greater tracking accuracy. By analysis of these perturbations, it is possible to deduce from the orbital data not only the average density, but also the variation of the density along the orbit. Moreover, a priori knowledge of the short-periodic perturbations is of concern in certain types of mission planning, such as in reconnaissance missions (where it is desirable to minimize altitude variations), in rendezvous planning, and in other multisatellite operations. For these reasons we concern ourselves with analyzing and predicting the complete trajectory, including the short-periodic perturbations.

The study of satellite trajectories under the influence of air drag has been the subject of several recent investigations.[2-4] We extend this work here, obtaining closed form solutions for the cases of large descent into a spherically symmetrical atmosphere with exponentially increasing density, and small descent under the influence of density that varies along the orbit as a result of diurnal variation or atmospheric oblateness.

Differential Equations for the Effects of Air Drag on Near-Circular Orbits

The first twelve equations follow the development given by Perkins.[5] The development is based on the assumption that the satellite velocity is close to horizontal, and its magnitude V can be approximately equated to the horizontal component. The vertical acceleration $\ddot{r}$ is the sum of the centrifugal and gravitational terms

$$\ddot{r} = (V^2/r) - (\mu/r^2) \tag{1}$$

where r is the magnitude of the radius vector from the center of the earth, and μ is the gravitational constant of the earth.

Let V_c be the velocity required for a circular orbit at the initial radius r_0. The departures in velocity V from V_c and in radius r from r_0 are small:

$$(V - V_c)/V_c \ll 1 \qquad (r - r_0)/r_0 \ll 1 \tag{2}$$

For a circular orbit, $V_c^2/r_0 = \mu/r_0^2$. Using this equality and expanding Eq. (1) in terms of departures of V from V_c and r from r_0, we have

$$\ddot{r} = 2\lambda(V - V_c) + \lambda^2(r - r_0) \tag{3}$$

where higher-order terms have been neglected in accordance with Eq. (2), and λ is the angular velocity for a circular orbit, $\lambda = V_c/r_0$.

The change in velocity from the initial velocity V_0 is caused by drag and by conversion of potential energy into kinetic energy as the altitude changes. Equating the change ΔE in the satellite energy to the work done by drag force D, we obtain

$$\Delta E = \frac{W}{g_{00}}\Delta\left[\frac{V^2}{2} - \frac{\mu}{r}\right] = \frac{W}{g_{00}}\left[V(V - V_0) + \frac{\mu}{r_2}(r - r_0)\right] = -\int_0^t D(\tau)Vd\tau \tag{4}$$

where W is the satellite weight, g_{00} is the standard acceleration of gravity, and τ is a dummy variable for time t. From Eq. (4), $V - V_0$ can be expressed as

$$V - V_0 = -\frac{g_\infty}{WV}\int_0^t \frac{C_D S V^3 \rho(\tau)d\tau}{2} - \frac{\mu}{Vr^2}(r - r_0) \tag{5}$$

where $\rho(t)$ is the ambient density encountered along the trajectory, C_D is the coefficient of drag, and S the satellite cross-sectional area. Perkins[5] introduces a drag parameter K, defined by

$$K \equiv g_{00}\rho_0 r_0^2/(W/C_D S) \tag{6}$$

where ρ_0 is a reference density. Neglecting second-order

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* Consulting Physicist, Missile and Space Division.
† Physicist, Missile and Space Division.

terms in the departures of V from V_c and the departures of r from r_0, Eq. (5) simplifies to

$$V - V_0 = -\frac{K\lambda^2}{2\rho_0}\int_0^t \rho(\tau)d\tau - \lambda(r - r_0) \qquad (7)$$

Introducing this $V - V_0$ into Eq. (3) yields

$$\ddot{r} = -\frac{K\lambda^3}{\rho_0}\int_0^t \rho(\tau)d\tau - \lambda^2(r - r_0) + 2\lambda(V_0 - V_c) \quad (8)$$

We now change the independent variable from t to ϕ (and from τ to θ), where

$$\phi = \lambda t \qquad (9)$$

The variable ϕ is measured in radians and can be regarded as the angular position of an infinitely heavy satellite, unaffected by drag, launched at r_0 with the horizontal velocity V_c. If primes denote differentiation with respect to ϕ, then

$$\lambda\int_0^t \rho(\tau)d\tau = \int_0^\phi \rho(\theta)d\theta \qquad r'' = \frac{\ddot{r}}{\lambda^2} \qquad (10)$$

We now rewrite Eq. (8), replacing $r - r_0$ by z for convenience:

$$z'' + z + \frac{K}{\rho_0}\int_0^\phi \rho(\theta)d\theta = 2\frac{V_0 - V_c}{\lambda} \qquad (11)$$

where $z(0) = 0$ (since initially $r = r_0$), and $z'(0)$ is the initial radial velocity component divided by λ. Differentiating with respect to ϕ, we obtain

$$z''' + z' + [K\rho(\phi)/\rho_0] = 0 \qquad (12)$$

where $z''(0) = 2(V_0 - V_c)/\lambda$, as specified by Eq. (11). This is the basic differential equation for radial descent from a near-circular orbit, obtained (for a circular orbit) in a very similar form by Perkins[5] as his Eq. (13).

We now extend Perkins' work to the question of the in-track effect of drag. This effect can be visualized as the in-track separation x of the actual satellite from an infinitely heavy satellite, unaffected by drag, placed at r_0 into a circular orbit. The rate of change of the separation is given by the difference in angular velocities projected on a circle with radius r_0:

$$\begin{aligned}\dot{x} &= r_0\left(\frac{V}{r} - \frac{V_c}{r_0}\right)\\ &= r_0\left[\frac{V_c}{r_0}\left(1 + \frac{V - V_c}{V_c} - \frac{r - r_0}{r_0}\right) - \frac{V_c}{r_0}\right] \qquad (13)\\ &= V - V_c - \lambda(r - r_0)\end{aligned}$$

Introducing the expression for $V - V_0$, Eq. (7), and using $z = r - r_0$, we obtain

$$\dot{x} = -\frac{K\lambda^2}{2\rho_0}\int_0^t \rho(\tau)d\tau - 2\lambda z + V_0 - V_c \qquad (14)$$

Changing the independent variable from t to ϕ, we have

$$\lambda\int_0^t \rho(\tau)d\tau = \int_0^\phi \rho(\theta)d\theta \qquad \frac{\dot{x}}{\lambda} = x' \qquad (15)$$

Equation (14) becomes

$$x' + \frac{K}{2\rho_0}\int_0^\phi \rho(\theta)d\theta + 2z = \frac{V_0 - V_c}{\lambda} \qquad (16)$$

This is a form of the differential equation for the in-track separation. To eliminate the explicit dependence on z, we first differentiate Eq. (16) with respect to ϕ:

$$x'' + [K\rho(\phi)/2\rho_0] + 2z' = 0 \qquad (17)$$

$$x''' + [K\rho'(\phi)/2\rho_0] + 2z'' = 0 \qquad (18)$$

Adding Eqs. (16) and (18), and introducing $z + z''$ from Eq. (11), we obtain

$$x''' + x' - \frac{3K}{2\rho_0}\int_0^\phi \rho(\theta)d\theta + \frac{K\rho'(\phi)}{2\rho_0} = -3\,\frac{V_0 - V_c}{\lambda} \qquad (19)$$

where $x(0) = 0$, $x'(0) = (V_0 - V_c)/\lambda$, and $x''(0) = -2z'(0) - K/2$. This last equality indicates the physical meaning of K; it is twice the reference drag deceleration (using radians of travel angle as the unit of time).

It should be pointed out that, in general, z still appears implicitly in this equation, since ρ is a function of z.

The particular integral of Eq. (12) can be formulated as the following convolution integral[6]:

$$z = -\frac{K}{\rho_0}\int_0^\phi \rho(\theta)[1 - \cos(\phi - \theta)]d\theta \qquad (20)$$

This can be shown by differentiating Eq. (20) three times with respect to ϕ. Adding the equations for z' and z''' will result in Eq. (12). A more general version of Eq. (20) has been obtained by Cook, King-Hele, and Walker.[7]

Likewise, the particular integral for x can be written as

$$x = -\frac{K}{2\rho_0}\int_0^\phi \rho(\theta)[4\sin(\phi - \theta) - 3(\phi - \theta)]d\theta \quad (21)$$

This again can be shown by successive differentiation of Eq. (21). Adding the equations for x'''' and x'' results in

$$x'''' + x'' - \frac{3K\rho(\phi)}{2\rho_0} + \frac{K\rho''(\phi)}{2\rho_0} = 0 \qquad (22)$$

which is Eq. (19) differentiated once.

The functions in Eqs. (20) and (21), respectively,

$$-\frac{K}{\rho_0}[1 - \cos(\phi - \theta)]$$

$$-\frac{K}{2\rho_0}\left[4\sin(\phi - \theta) - 3(\phi - \theta)\right]$$

describe the effects of the density at θ on the position of the satellite at a later travel angle ϕ. Readers familiar with the concept of the convolution integral will note that these functions can be regarded as responses to a density impulse.

The initial orbit eccentricity (and location of perigee) can be expressed in terms of $z'(0)$ and $z''(0)$. With this in mind, an equation for the radius r as a function of ϕ, given $z'(0)$ and $z''(0)$, in the absence of drag, is sought. This can be obtained by solving Eq. (12) with $K = 0$. The solution is

$$z = z''(0)(1 - \cos\phi) + z'(0)\sin\phi \qquad (23)$$

We compare this solution with the equation of an elliptic orbit for small eccentricities:

$$r = \frac{a(1 - e^2)}{1 + e\cos(\phi - \phi_0)} \approx a[1 - e\cos(\phi - \phi_0)] \quad (24)$$

where a is the semimajor axis, e the eccentricity, and the perigee is located at $\phi = \phi_0$. Now since $z = r - r_0$, we have

$$\begin{aligned}z &= a[1 - e\cos(\phi - \phi_0)] - a[1 - e\cos\phi_0]\\ &= ae[\cos\phi_0(1 - \cos\phi) - \sin\phi_0\sin\phi] \qquad (25)\end{aligned}$$

Comparing Eq. (25) with Eq. (23) results in

$$(ae)^2 = [z''(0)]^2 + [z'(0)]^2 \qquad (26)$$

$$\tan\phi_0 = -z'(0)/z''(0) \qquad (27)$$

Moreover, since a represents the average radius in Eq. (24), we have, in view of Eq. (23),

$$a = r_0 + z_{\text{av}} = r_0 + z''(0) \qquad (28)$$

Satellite Trajectories in a Spherically Symmetrical Atmosphere

Constant Density

In this subsection, we are concerned with the case of small descent, i.e., z much smaller than the density scale height H, and, therefore, in a spherically symmetrical atmosphere, the density is approximately constant, $\rho(\phi)/\rho_0 = 1$. With this assumption, Eq. (12) for z becomes

$$z''' + z' + K = 0 \tag{29}$$

The solution is the sum of the complementary function, which has been obtained in Eq. (23) as the solution of the homogenous equation ($K = 0$), and the particular integral

$$-K\int_0^\phi [1 - \cos(\phi - \theta)]d\theta$$

thus

$$\begin{aligned} z &= z'(0)\sin\phi + z''(0)(1 - \cos\phi) - K(\phi - \sin\phi) \\ &= -K(\phi) + [K + z'(0)]\sin\phi + z''(0)(1 - \cos\phi) \end{aligned} \tag{30}$$

The particular integral $-K(\phi - \sin\phi)$, which has been obtained by Perkins[5] and Karrenberg et al.,[2] indicates that drag causes a descent at a constant rate with a superimposed oscillation of constant amplitude.

With the assumption of constant density, Eq. (22), for the in-track separation, becomes

$$x'''' + x'' - (3K/2) = 0 \tag{31}$$

and the solution is

$$x = \tfrac{3}{4}K\phi^2 - 2[K + z'(0)](1 - \cos\phi) - 2z''(0)(\tfrac{3}{4}\phi - \sin\phi) \tag{32}$$

The particular integral in the solution $\frac{3}{4}K\phi^2 - 2K(1 - \cos\phi)$, which has been obtained by Karrenberg et al.,[2] indicates that an actual satellite will initially lag behind the theoretical no-drag satellite; then at about 105° of travel,[8] it will catch up (at a lower altitude, however) and finally will lead the no-drag satellite.

It is clear from inspection of Eqs. (30) and (32) that the amplitudes of the oscillatory terms depend both on K and on departure of the initial velocity from V_0. In particular, examine Eqs. (30) and (32) for the special case when

$$z'(0) = -K \qquad z''(0) = 0 \tag{33}$$

The equations for z and x are then

$$z = -K\phi \tag{34}$$

$$x = {}^3K\phi^2 \tag{35}$$

which indicate a steady descent, without oscillations. This type of spiral has been analyzed by Bacon.[8] We want to bring out the fact that such a trajectory‡ with no oscillating terms is possible, given the right initial conditions. At the same time, the drag effects are still given by the same particular integrals with their oscillatory terms.

The periodic oscillations, due to drag, in semimajor axis, eccentricity, and argument of perigee have been discussed by Sterne.[9]

Exponentially Increasing Density, Near-Circular Orbit

We continue to assume a spherically symmetrical atmosphere, but we take into account changes in density due to changes in satellite altitude, both the descent resulting from the drag, and the altitude variations due to initial orbit eccentricity and due to oscillatory terms induced by drag. We assume an exponentially changing density

$$\rho(z) = \rho_0 e^{-z/H} \tag{36}$$

where ρ_0 is the density at the altitude corresponding to r_0 and where H is the density scale height. Introducing this relation into Eq. (12), we have

$$z''' + z' + Ke^{-z/H} = 0 \tag{37}$$

We introduce a dimensionless altitude parameter ζ and a dimensionless drag parameter κ:

$$\zeta \equiv z/H \qquad \kappa \equiv K/H \tag{38}$$

Equation (37) is restated as

$$\zeta''' + \zeta' + \kappa e^{-\zeta} = 0 \tag{39}$$

This is a nonlinear differential equation for which an exact solution has not been obtained. This equation has been set up on an analog computer for values of κ ranging from 0.002 to 0.03.[8] The solutions indicated descent at a gradually increasing rate with superimposed oscillations. The amplitude of oscillations decreases with the descent. The analog computer solutions prompted attempts to obtain approximate analytic solutions of Eq. (39), and the following equation has been derived:

$$\zeta = \ln(1 - \kappa\phi) + (1 - \kappa\phi)^{1/2}(B\sin\phi + C\cos\phi) + [A/(1 - \kappa\phi)] \tag{40}$$

This expression constitutes an approximate solution when terms are neglected which are of second-order or higher in powers of the small quantities defined by the following restrictions:

$$\begin{aligned} &\kappa/(1 - \kappa\phi) \ll 1 \qquad && A/(1 - \kappa\phi) \ll 1 \\ &B(1 - \kappa\phi)^{1/2} \ll 1 && C(1 - \kappa\phi)^{1/2} \ll 1 \end{aligned} \tag{41}$$

To demonstrate this, we differentiate ζ [Eq. (40)] three times and expand $\kappa e^{-\zeta}$:

$$\zeta' = -\frac{\kappa}{1 - \kappa\phi} + (1 - \kappa\phi)^{1/2}(B\cos\phi - C\sin\phi) - \frac{\kappa}{2}\frac{(B\sin\phi + C\cos\phi)}{(1 - \kappa\phi)^{1/2}} + \frac{\kappa A}{(1 - \kappa\phi)^2} \tag{42}$$

$$\zeta'' = -\frac{\kappa^2}{(1 - \kappa\phi)^2} + (1 - \kappa\phi)^{1/2}(-B\sin\phi - C\cos\phi) - \frac{\kappa(B\cos\phi - C\sin\phi)}{(1 - \kappa\phi)^{1/2}} - \frac{\kappa^2}{4}\frac{(B\sin\phi + C\cos\phi)}{(1 - \kappa\phi)^{3/2}} + \frac{2\kappa^2 A}{(1 - \kappa\phi)^3} \tag{43}$$

$$\zeta''' = -\frac{2\kappa^3}{(1 - \kappa\phi)^3} + (1 - \kappa\phi)^{1/2}(-B\cos\phi + C\sin\phi) + \frac{3\kappa}{2}\frac{(B\sin\phi + C\cos\phi)}{(1 - \kappa\phi)^{1/2}} - \frac{3\kappa^2}{4}\frac{(B\cos\phi - C\sin\phi)}{(1 - \kappa\phi)^{3/2}} - \frac{3\kappa^3}{8}\frac{(B\sin\phi + C\cos\phi)}{(1 - \kappa\phi)^{5/2}} + \frac{6\kappa^3 A}{(1 - \kappa\phi)^4} \tag{44}$$

$$\begin{aligned} \kappa e^{-\zeta} &= \frac{\kappa}{1 - \kappa\phi}\exp\left[-(1 - \kappa\phi)^{1/2} \times (B\sin\phi + C\cos\phi) - \frac{A}{1 - \kappa\phi}\right] \\ &= \frac{\kappa}{1 - \kappa\phi} - \frac{\kappa(B\sin\phi + C\cos\phi)}{(1 - \kappa\phi)^{1/2}} - \frac{\kappa A}{(1 - \kappa\phi)^2} + \frac{\kappa}{2}\left[B\sin\phi + C\cos\phi + \frac{A}{(1 - \kappa\phi)^{3/2}}\right]^2 + \ldots \end{aligned} \tag{45}$$

We now form the sum $\zeta''' + \zeta' + \kappa e^{-\zeta}$. Most terms cancel

‡ Bacon's exact analysis indicates that the applied thrust-to-mass ratio must change in the same proportion as the local gravitational field strength.

out. The remaining terms are of the third-order or higher in powers of the small quantities defined by Eq. (41). These uncancelled terms correspond to terms that would appear in an exact solution for ζ and that are of second-order (or higher) in powers of the same small quantities.

The constants A, B, and C depend on the initial conditions and are determined by Eqs. (40, 42, and 43) for $\phi = 0$. Since $\zeta(0) = 0$, we have

$$A = -C = \frac{\zeta''(0) + 2\kappa^2 + \kappa\zeta'(0)}{1 + 15\kappa^2/4} \approx \zeta''(0) \tag{46}$$

$$B = \kappa + \zeta'(0) - \frac{3\kappa[\zeta''(0) + 2\kappa^2 + \kappa\zeta'(0)]}{2(1 + 15\kappa^2/4)} \approx \kappa + \zeta'(0) \tag{47}$$

Turning our attention to the in-track effect, we define

$$\xi \equiv x/H \tag{49}$$

as the dimensionless in-track perturbation. After dividing by H, Eq. (16) can be restated as

$$\xi' + \frac{\kappa}{2\rho_0}\int_0^\phi \rho(\theta)d\theta + 2\zeta = \xi'(0) = \frac{V_0 - V_c}{H\lambda} \tag{50}$$

and in view of the assumption of exponentially changing density [see Eq. (36)]:

$$\xi' + \frac{\kappa}{2}\int_0^\phi e^{-\zeta(\theta)}\, d\theta + 2\zeta = \xi'(0) \tag{51}$$

To evaluate the integral, we expand $e^{-\zeta}$, as in Eq. (45),

$$\begin{aligned}
\frac{\kappa}{2}\int_0^\phi e^{-\zeta(\theta)}\, d\theta &= \frac{\kappa}{2}\int_0^\phi \frac{\exp[-(1-\kappa\theta)^{1/2}(B\sin\theta + C\cos\theta) - A(1-\kappa\theta)^{-1}]d\theta}{1-\kappa\theta} \\
&\approx \frac{\kappa}{2}\int_0^\phi \left[\frac{1}{1-\kappa\theta} - \frac{B\sin\theta + C\cos\theta}{(1-\kappa\theta)^{1/2}} - \frac{A}{(1-\kappa\theta)^2}\right]d\theta \\
&= \left\{-\frac{\ln(1+\kappa\theta)}{2} + \frac{\kappa}{2}\left[\frac{B\cos\theta - C\sin\theta}{(1-\kappa\theta)^{1/2}} - \frac{\kappa}{2}\int \frac{B\cos\theta - C\sin\theta}{(1-\kappa\theta)^{3/2}}d\theta\right] - \frac{A}{2(1-\kappa\theta)}\right\}_0^\phi \\
&= -\frac{\ln(1-\kappa\phi)}{2} + \frac{\kappa}{2}\left[\frac{B\cos\phi - C\sin\phi}{(1-\kappa\phi)^{1/2}} - B - \frac{\kappa}{2}\int_0^\phi \frac{B\cos\theta - C\sin\theta}{(1-\kappa\theta)^{3/2}}d\theta\right] - \frac{A}{2(1-\kappa\phi)} + \frac{A}{2}
\end{aligned} \tag{52}$$

Equation (40) provides a solution to Eq. (39) accurate within a few percent for most of satellite lifetime even for a drag parameter as high as $\kappa = 0.03$ (which corresponds to a lifetime of only about five orbits). The eccentricity is assumed 0.001 or less, with $A \leqslant 0.03$ and $B \leqslant 0.15$.

The first term of Eq. (40), $\ln(1 - \kappa\phi)$, represents a steady logarithmic descent. We note that, for $\phi \to 1/\kappa$, the term becomes minus infinity. Thus $1/\kappa$ can be regarded as the satellite lifetime (in radians of travel angle). When a large fraction of the satellite lifetime has elapsed, $\ln(1 - \kappa\phi)$ become the predominant part of the solution.

The second term of Eq. (40) represents a damped oscillation. When the steady logarithmic descent predominates [$\zeta \approx \ln(1 - \kappa\phi)$], we have

$$(1-\kappa\phi)^{1/2}(B\sin\phi + C\cos\phi) \approx e^{\zeta/2}(B\sin\phi + C\cos\phi) \tag{48}$$

We note that the damping rate is such that, after a descent of one scale height ($\zeta = -1$), the oscillatory terms will be damped out to only $e^{-1/2} = 0.6$ of their initial amplitudes. The amplitudes B and C of the oscillatory terms are linked to the departures of the initial orbit from a circular orbit [see Eqs. (46) and (47)]. It will be noted that for an initially circular orbit, $\zeta'(0) = \zeta''(0) = 0$, an oscillation ensues of amplitude $B \approx \kappa$, as we established before. The interesting thing here is that this oscillatory term, $\kappa \sin\phi$, which is due to drag, is damped out in the same fashion as the oscillatory terms due to the orbit eccentricity. The damping out of the oscillations agrees qualitatively with conclusions by Zee[3§] and Forster and Baker.[4]

The third term in Eq. (40) somewhat modifies the rate of steady descent given by the first term and is linked to the fact that, when $\zeta''(0)$ is positive, the initial semimajor axis a is larger than the initial radius r_0 [see Eq. (28)], and the average drag along the orbit (and therefore the average rate of decay) is smaller than at the initial point.

§ According to Zee's analysis, the radial oscillations are damped out in such a way that, after a descent of one scale height H, their amplitude decreases by a fraction $3H/2r_0$, which is roughly 2%, as compared to a decrease of 40% obtained here.

where integration was performed by parts. The integral

$$\frac{\kappa^2}{2}\int_0^\phi \frac{B\cos\theta - C\sin\theta}{(1-\kappa\theta)^{3/2}}d\theta$$

can be evaluated by continuous integration by parts, leading to terms in increasing powers of κ and a remainder integral. Since the first term of the series is

$$\frac{\kappa^2}{4}\frac{B\sin\phi + C\cos\phi}{(1-\kappa\phi)^{3/2}}$$

we neglect the series compared to other terms in Eq. (52).

To obtain ξ, we integrate Eq. (51):

$$\begin{aligned}
\xi &= \int_0^\phi \Bigg[\xi'(0) + \frac{\ln(1-\kappa\theta)}{2} - \frac{\kappa}{2}\frac{B\cos\theta - C\sin\theta}{(1-\kappa\theta)^{1/2}} + \frac{\kappa B}{2} + \frac{A}{2(1-\kappa\theta)} - \\
&\qquad \frac{A}{2} - 2\ln(1-\kappa\theta) - 2(1-\kappa\theta)^{1/2}(B\sin\theta + C\cos\theta) - 2\frac{A}{1-\kappa\theta}\Bigg]d\theta \\
&= \int_0^\phi \Bigg[\xi'(0) - \frac{3}{2}\ln(1-\kappa\theta) + \frac{\kappa}{2}\frac{B\cos\theta - C\sin\theta}{(1-\kappa\theta)^{1/2}} + \frac{\kappa B}{2} - \frac{3A}{2(1-\kappa\theta)} - \\
&\qquad \frac{A}{2} - 2(1-\kappa\theta)^{1/2}(B\sin\theta + C\cos\theta)\Bigg]d\theta \\
&= \xi'(0)\phi + \frac{3}{2\kappa}[(1-\kappa\phi)\ln(1-\kappa\phi) + \kappa\phi] + \frac{\kappa}{2}\frac{B\sin\phi - C\cos\phi}{(1-\kappa\phi)^{1/2}} + \frac{\kappa C}{2} - \\
&\qquad \frac{\kappa^2}{4}\int_0^\phi \frac{B\sin\theta - C\cos\theta}{(1-\kappa\theta)^{3/2}}d\theta + \frac{\kappa B\phi}{2} +
\end{aligned} \tag{53}$$

$$\frac{3}{2\kappa} A \ln(1 - \kappa\phi) - \frac{A\phi}{2} - 2(1 - \kappa\phi)^{1/2} (-B \cos\phi + C \sin\phi) - 2B - \kappa \int_0^\phi \frac{-B \cos\theta + C \sin\theta}{(1 - \kappa\theta)^{1/2}} d\theta \quad (53)$$

Introducing Eqs. (46) and (47), and since $\xi'(0) = \zeta''(0)/2$, we have

$$\xi \approx (3/2\kappa)[(1 - \kappa\phi) \ln(1 - \kappa\phi) + \kappa\phi + \zeta''(0) \ln(1 - \kappa\phi)] + 2(1 - \kappa\phi)^{1/2} \times [(\zeta'(0) + \kappa) \cos\phi + \zeta''(0) \sin\phi] - [\zeta'(0) + \kappa][2 - (\kappa\phi/2)] \quad (54)$$

where higher-order terms have been neglected.

It may be noted that, for $\kappa \to 0$, the expression

$$\frac{3\zeta''(0) \ln(1 - \kappa\phi)}{2\kappa} \to - \frac{3\zeta''(0)\phi}{2}$$

Thus, for $\kappa = 0$, Eq. (54) reverts to the no-drag solution, as is to be expected.

Two points can be made concerning the accuracy of Eqs. (40) and (54). First, the changes of orbital parameters due to the descent are neglected compared to the change in density. These changes are very small compared to the density increase. A much more severe limitation is the assumption of a spherically symmetrical exponential density model. It is true that the exponential variation with altitude is a good approximation above a given point on the earth at a given time in the intermediate satellite altitudes (from 130 to 300 naut miles), since the temperature, and therefore the scale height H, varies only slowly with altitude. However, above 130 naut miles the diurnal changes in density will be quite pronounced (see next section), and the spherical symmetry fails, except in an approximately twilight¶ orbit (when the satellite would not cross the density maximum or density minimum regions). Above 300 naut miles the effects of solar radiation have to be considered.[1] At the low satellite altitudes of 80 to 130 naut miles, the assumption of spherical symmetry is quite reasonable, but the exponential model (constant H) is rather inaccurate.

It should be stressed that because of the gravitational model, Eq. (1), the orbital effects of earth gravitational oblateness, such as change of inclination and rotation of the orbit, are not included in the solution.

Effects of a Spherically Nonsymmetrical Atmosphere

In this section on nonsymmetrical atmosphere, the change in density due to the descent itself is neglected. In what follows, the particular integrals of drag effects are presented which are complete solutions only for initially circular orbits. Thus, z and x represent the orbital perturbations due to drag.

¶ The maximum density point lags the subsolar (noon) point by about 2 hr.[11] The region of approximately constant density (for a circular orbit) is removed from the true twilight region by this amount.

Effects of the Diurnal Bulge

For altitudes of 130 naut miles and higher, the density on the sunlit side of the earth is considerably greater than that on the dark side, as is discussed by Jacchia.[10] According to the recent atmospheric model by Harris and Priester,[11] the peak daytime density is greater than the minimum night-time density by a factor of 1.7 at an altitude of 130 naut miles, and the ratio increases to a factor of 3.9 at 300 naut miles [for 10.7-cm solar flux of 100×10^{-22} (w/m^2)/cps]. The effects of the diurnal bulge on orbital parameters have been investigated by Davies.[12]

In this subsection we shall analyze the effects of the diurnal bulge on the trajectory of a satellite, assuming a rather severe simplification of the density variation along the orbit [Eq. (55)]. Later, we will justify the simplification to some extent.

We assume that the density along the flight path can be expressed as

$$\rho(\phi) = \rho_0[1 + D \cos(\phi - \phi_m)] \quad (55)$$

where ρ_0, the reference density (not the initial density) is the average density around the earth, the density maximum is located at ϕ_m, and the parameter D is given by

$$D = \frac{\rho_{\max} - \rho_{\min}}{\rho_{\max} + \rho_{\min}} \quad (56)$$

Introducing the density relation into Eq. (20),

$$z = -K \int_0^\phi [1 + D \cos(\theta - \phi_m)][1 - \cos(\phi - \theta)]d\theta = -K\left[\phi - \sin\phi + \frac{D}{2} \cos\phi_m \sin\phi + D \sin\phi_m(1 - \cos\phi) - \frac{D}{2}\phi \cos(\phi - \phi_m)\right] \quad (57)$$

The expression $-K(\phi - \sin\phi)$ represents the descent (with oscillations) due to the average density around the orbit. The next two terms represent oscillations of constant amplitude. The last term is an oscillation building up as $\phi/2$. This is a relatively important effect. For an initially circular orbit, the effect is of an increasing eccentricity, which can be visualized readily comparing this term with the expression for the radius in a near-circular orbit, $r \approx a[1 - e \cos(\phi - \phi_m)]$. The apogee is at the density peak (ϕ_m) and the perigee at the density minimum ($\phi_m + \pi$) of the original circular orbit. The magnitude of the eccentricity is $KD\phi/2a$. Still another way of looking at this eccentricity effect is to note that at the density maximum, $\phi = \phi_m$, the altitude drop is $K\phi(1 - D/2)$, and at the density minimum, $\phi = \phi_m + \pi$, the altitude drop is $K\phi(1 + D/2)$, where only the first and last terms of the solution are taken into account. Thus, from Eq. (56), the altitude drop at the nascent perigee is greater than that at the nascent apogee by the factor $(3\rho_{\max} + \rho_{\min})/(3\rho_{\min} + \rho_{\max})$.**

To obtain the in-track effect x, we integrate Eq. (21) as

$$x = -\frac{K}{2} \int_0^\phi [1 + D \cos(\theta - \phi_m)][4 \sin(\phi - \theta) - 3(\phi - \theta)]d\theta = \frac{K}{2} \left\{\frac{3}{2} \phi^2 - 4(1 - \cos\phi) - D\phi[2 \sin(\phi - \phi_m) - 3 \sin\phi_m] - 2D \sin\phi_m \sin\phi - 3D \cos(\phi - \phi_m) + 3D \cos\phi_m\right\} \quad (58)$$

** If one wishes to maintain the circular orbit by impulsive tangential thrusting, the impulse at the density maximum should be greater than that at the density minimum by the same factor.

It should be stressed here that the change in density due to the descent $-z$ has not been taken into account. The build-up of eccentricity will gradually slow down as the situation of the satellite encountering approximately equal density throughout the orbit is approached, since it crosses the diurnal bulge at a higher altitude than its altitude in the region of nocturnal density minimum.

For heavy satellites at relatively high altitudes (small κ) the descent will be small even for a period of a few months. In this case the accuracy of the foregoing analysis is limited by the motions of the nascent perigee because of the earth's gravitational oblateness and the shift of the diurnal bulge due to the earth's motion around the sun.

Still another point should be made concerning the accuracy of this analysis. The sinusoidal variation of density, Eq. (55), will not match too accurately the actual density along the orbit, since the diurnal bulge is rather sharp and the night-time minimum rather flat.[11] Partial justification of this simple density model lies in the fact that the effects of harmonics are weak compared to the "resonant" term $D \times \cos(\phi - \phi_m)$. But for higher accuracy, D should be determined by the Fourier analysis of the density profile along the orbit rather than by Eq. (56).

The numerical example given later applies to an orbit that passes through the diurnal bulge and the night-time minimum. The density maxima and minima in an actual orbit can be determined by means of the plots and tables provided by Harris and Priester. A geometrical analysis of the diurnal variation has been carried out by Jacchia.[13]

Analysis of an exactly circular orbit is of only very limited interest.[4] The authors advocate the application of the orbital mechanics of an exactly circular orbit to cases of small eccentricity as has been done earlier in this paper. Thus, Eqs. (57) and (58) can be applied to the case where D, i.e., ρ_{min} and ρ_{max}, is determined by the combined geometry of the diurnal bulge and a small orbit eccentricity.

Effects of Atmospheric Oblateness

Satellites traveling in other-than-equatorial orbits will encounter a variation in density linked to the fact that surfaces of constant altitude are roughly oblate spheroids rather than spheres, following the lines of the oblate earth. The effect of atmospheric oblateness on the decay rates of apogee and perigee altitudes has been studied by King-Hele.[14] As before, we concern ourselves with the descent and in-track effect along the orbit.

The oblateness will produce two local density minima along the orbit at points closest to the poles and two maxima at the equator crossings. It should be pointed out that this effect might be modified (partially counterbalanced) by the polar density bulge reported by Groves.[15] The main effect in a gross approximation can be regarded as a sinusoidal variation with an angular velocity twice that of the satellite. The total density along the orbit is then

$$\rho(\phi) = \rho_0[1 + E\cos 2(\phi - \phi_p)] \qquad (59)$$

where ρ_0 is the average density, the density peak occurs at $\phi = \phi_p$, and E is the difference between the maximum and minimum densities divided by their sum, as in Eq. (56).

Using this density model, Eq. (20) yields the following solution:

$$z = -K\left[\phi - \sin\phi - \frac{2E}{3}\sin 2\phi_p\cos\phi + \frac{E}{2}\sin 2\phi_p - \frac{E}{6}\sin 2(\phi - \phi_p) + \frac{E}{3}\cos 2\phi_p\sin\phi\right] \qquad (60)$$

This equation contains terms of the following kind: the expression $-K(\phi - \sin\phi)$, which is linked to the average density along the orbit; oscillatory terms with a characteristic period equal to the period of revolution and with a combined amplitude of $(KE/3)(\cos^2 2\phi_p + 4\sin^2 2\phi_p)^{1/2}$; and an oscillatory term with a period equal to half of the period of revolution with an amplitude of $KE/6$.

For the in-track effect, Eq. (21) gives the following result:

$$x = K\left\{\frac{3\phi^2}{4} - 2(1 - \cos\phi) + \frac{3E\phi}{4}\sin 2\phi_p - \frac{4E}{3}\sin 2\phi_p\sin\phi - \frac{7E}{24}[\cos 2\phi_p - \cos 2(\phi - \phi_p)] + \frac{2E}{3}\cos 2\phi_p(1 - \cos\phi)\right\} \qquad (61)$$

It may be of interest to note that Eqs. (60) and (61) can be regarded as approximate drag equations in the case of a nonspherical satellite that is stabilized in inertial space. During an orbit, such a satellite will twice exhibit its maximum cross section and twice its minimum cross section. Equations (60) and (61) can then apply either under the assumption of constant density or oblate atmosphere when E is interpreted as the difference between the maximum and minimum tangential deceleration encountered along the orbit, divided by their sum.

Effects of the Higher Harmonics

We consider the Fourier-series representation of the density along the orbit

$$\rho(\phi) = \rho_0\left[1 + \sum_{n=1}^{\infty} a_n\cos n(\phi - \phi_n)\right] \qquad (62)$$

and the influence of the individual terms $a_n\cos n(\phi - \phi_n)$ on the satellite trajectory. Perturbations in semimajor axis and eccentricity when density is represented by Fourier series have been presented by Batrakov and Proskurin.[16] Using the Fourier representation, the solution for z is

$$z = -K\left[\phi - \sin\phi - \frac{a_1\phi}{2}\cos(\phi - \phi_1) + \left(\frac{a_1}{2}\cos\phi_1 + \sum_{n=2}^{\infty}\frac{a_n}{n^2 - 1}\cos n\phi_n\right)\sin\phi - \left(a_1\sin\phi_1 + \sum_{n=2}^{\infty}\frac{a_n n}{n^2 - 1}\sin n\phi_n\right)\cos\phi + a_1\sin\phi_1 - \sum_{n=2}^{\infty}\left(\frac{a_n}{n^3 - n}\sin n(\phi - \phi_n) - \frac{a_n}{n}\sin n\phi_n\right)\right] \qquad (63)$$

The coefficient a_1 of the diurnal bulge (and eccentricity) term and the coefficient a_2 of the oblateness term usually are expected to be much higher than the coefficients of higher terms. Thus, it is expected that the effects of higher terms on the trajectory are quite small, particularly since the higher harmonics appear in the equation for z with amplitudes $a_n/(n^3 - n)$.

Numerical Examples

We consider the case of a space station with $W/C_DS = 15$ psf, orbiting at a reference altitude of 200 naut miles. The equatorial radius of the earth is taken to be 3444 naut miles. We have then $r_0 = 3444 + 200 = 3644$ naut miles $= 2.214 \times 10^7$ ft.

Calculation of K and κ, Using 1962 U. S. Standard Atmosphere[17]

For this case we have $\rho_0 g_{00} = 6.505 \times 10^{-13}$ lb/ft³. Taking the slope of the log density vs altitude curve, we obtain $H = 33$ naut miles $= 2.0 \times 10^5$ ft. Therefore

$$K = \frac{\rho_0 g_{00} r_0^2}{W/C_DS} = 21.3 \text{ ft}$$

$$\kappa = K/H = 1.06 \times 10^{-4}$$

This value of κ corresponds to a lifetime of $L = 1/\kappa = 0.94 \times 10^4$ rad = 1500 revolutions $\approx$ 94 days.

Calculation of *K*, *D*, and Eccentricity, Using Harris and Priester Atmosphere[11]

In this case the diurnal variation of density is taken into account. The Harris and Priester tables for a 10.7 cm solar flux of 100×10^{-22} (w/m²)/cps are used.

We have:

$$\rho_{\max} g_{00} = 2.574 \times 10^{-16} \text{ lb/ft}^3$$

$$\rho_{\min} g_{00} = 0.664 \times 10^{-16} \text{ lb/ft}^3$$

Therefore,

$$\rho_0 g_{00} = \tfrac{1}{2}(\rho_{\max} g_{00} + \rho_{\min} g_{00}) = 1.619 \times 10^{-13} \text{ lb/ft}^3$$

$$K = \frac{\rho_0 g_{00}\, r_0^2}{W/C_D S} = 5.29 \text{ ft}$$

$$D = \frac{\rho_{\max} g_{00} - \rho_{\min} g_{00}}{\rho_{\max} g_{00} + \rho_{\min} g_{00}} = 0.59$$

After ten days ($\phi \approx 160$ revolutions ≈ 1000 rad), the altitude difference between the nascent apogee and the nascent perigee of an initially circular orbit would build up to $KD\phi$ = 3120 ft $\approx$ 0.5 naut miles, which corresponds to an eccentricity of $e = KD\phi/2a = KD\phi/2r_0 = 0.00007$.

Calculation of *E*, Using 1962 U. S. Standard Atmosphere

The difference between the equatorial and polar radii of the earth is 11.5 naut miles. For a satellite in a circular polar orbit with a mean altitude of 200 naut miles, we have

maximum altitude = 205.75 naut miles
minimum altitude = 194.25 naut miles

$$\rho_{\min} g_{00} = 5.471 \times 10^{-13} \text{ lb/ft}^3$$

$$\rho_{\max} g_{00} = 7.757 \times 10^{-13} \text{ lb/ft}^3$$

$$E = \frac{\rho_{\max} g_{00} - \rho_{\min} g_{00}}{\rho_{\max} g_{00} + \rho_{\max} g_{00}} = 0.17$$

For this case the amplitudes of the oblateness terms in Eq. (60), which are of the order of KE, are negligible.

Conclusions

Equations have been presented for the radial and in-track effects of air drag on satellites in near-circular orbits. It is thought that the expressions presented here can be useful for mission planning purposes. Also, the analysis can be used in orbit-determination computer programs to apply corrections to actual tracking observations in order to fit the data to an unperturbed orbit.

It can be concluded from the analysis that, in the case of a descent into a spherically symmetrical atmosphere with exponentially increasing density, the oscillatory terms due either to the initial small eccentricity or to drag effects tend to die out. Their amplitude decreases to 0.6 of the initial amplitude after a descent of one scale height.

It has been shown that the effects of the diurnal density variations can be quite pronounced.

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